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GEODESIC DISTANCE FOR RIGHT INVARIANT SOBOLEV METRICS OF FRACTIONAL ORDER ON THE DIFFEOMORPHISM GROUP

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ABSTRACT. We study Sobolev-type metrics of fractional order $s \geq 0$ on the group $\mathrm{Diff}_c(M)$ of compactly supported diffeomorphisms of a manifold M. We show that for the important special case $M = S^1$ the geodesic distance on $\mathrm{Diff}_c(S^1)$ vanishes if and only if $s \leq \frac{1}{2}$. For other manifolds we obtain a partial characterization: the geodesic distance on $\mathrm{Diff}_c(M)$ vanishes for $M = \mathbb{R} \times N, s < \frac{1}{2}$ and for $M = S^1 \times N, s \leq \frac{1}{2}$, with N being a compact Riemannian manifold. On the other hand the geodesic distance on $\mathrm{Diff}_c(M)$ is positive for $\dim(M) = 1, s > \frac{1}{2}$ and $\dim(M) \geq 2, s \geq 1$.

For $M = \mathbb{R}^n$ we discuss the geodesic equations for these metrics. For $n = 1$ we obtain some well known PDEs of hydrodynamics: Burgers' equation for $s = 0$, the modified Constantin-Lax-Majda equation for $s = \frac{1}{2}$ and the Camassa-Holm equation for $s = 1$.

1. INTRODUCTION

In the seminal paper [1] Arnold showed that the incompressible Euler equations can be seen as geodesic equations on the group of volume preserving diffeomorphisms with respect to the L^2-metric. This interpretation was extended to other PDEs used in hydrodynamics, some of which are: Burgers' equation as the geodesic equation for $\mathrm{Diff}_c(\mathbb{R})$ with the L^2-metric, Camassa-Holm equation for $\mathrm{Diff}_c(\mathbb{R})$ with the H^1-metric or KdV for the Virasoro-Bott group with the L^2-metric in [21]. Recently it was shown by Wunsch [28] that the modified Constantin-Lax-Majda equation (mCLM) is the geodesic equation on the homogeneous space $\mathrm{Diff}(S^1)/S^1$ with respect to the homogeneous $\dot{H}^{1/2}$-metric.

The geometric interpretation was used by Ebin and Marsden in [6] to show the well-posedness of the Euler equations. These techniques were expanded and applied to other equations, see e.g. [4, 5, 10, 11].

The interpretation of a PDE as the geodesic equation on an infinite dimensional manifold opens up a variety of geometrical questions, which may be asked about the manifold. What is the curvature of this manifold? Do geodesics have conjugate points? Do there exist totally geodesic submanifolds? The question we want to concentrate upon in this paper, that of geodesic distance, has a very simple answer

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in finite dimensions but shows more nuances in infinite dimensions. The geodesic distance between two points is defined as the infimum of the pathlength over all paths connecting the two points. In finite dimensions, because of the local invertibility of the exponential map, this distance is always positive and the topology of the resulting metric space is the same as the manifold topology.

However, in infinite dimensions, this does not always hold: the induced geodesic distance for weak Riemannian metrics on infinite dimensional manifolds may vanish. This surprising fact was first noticed for the L^2-metric on shape space $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1)$ in [19, 3.10]. Here $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1)$ denotes the orbifold of all immersions $\operatorname{Imm}(S^1,\mathbb{R}^2)$ of S^1 into $\mathbb{R}^2$ modulo reparametrizations. In [18] it was shown that this result holds for the general shape space $\operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ for any compact manifold M and Riemannian manifold N, and also for the right invariant L^2-metric (or equivalently Sobolev-type metric of order zero) on each full diffeomorphism group with compact support $\operatorname{Diff}_c(N)$. In particular, since Burgers' equation is related to the geodesic equation of the right invariant L^2-metric on $\operatorname{Diff}_c(\mathbb{R}^1)$, it implies that solutions of Burgers' equation are critical points of the length functional, but they are not length-minimizing. A similar result was shown for the KdV equation in [2].

On the other hand it was shown in [18, 3] that for Sobolev-type metrics on the diffeomorphism group of order one or higher the induced geodesic distance is positive. This naturally leads to the question whether one can determine the Sobolev order where this change of behavior occurs. This paper gives a complete answer in the case of $M = S^1$ and a partial answer for other manifolds. Our main result is:

Theorem (Geodesic distance)**.** *Let M be a Riemannian manifold and $\operatorname{Diff}_c(M)$ the group of compactly supported diffeomorphisms of M.*

(1) *The geodesic distance for the fractional order Sobolev type metric H^s on $\operatorname{Diff}_c(M)$ vanishes for*
  - *$0 \leq s < \frac{1}{2}$ and M a Riemannian manifold that is the product of $\mathbb{R}$ with a compact manifold N, i.e., $M = \mathbb{R} \times N$.*
  - *$s = \frac{1}{2}$ and M a Riemannian manifold that is the product of S^1 with a compact manifold N, i.e., $M = S^1 \times N$.*

(2) *For $\dim(M) = 1$ the induced geodesic distance is positive for $\frac{1}{2} < s$ and for general $\dim(M) \geq 2$ the geodesic distance is positive for $1 \leq s$.*

Let us now briefly outline the structure of this work. In Section 2 we review the basic definition of Sobolev-type metrics on diffeomorphism groups. For a compact Riemannian manifold M (see 2.4) we consider the Sobolev metric H^s of order s on the Lie algebra of vector fields and the induced right invariant metric on the diffeomorphism group $\operatorname{Diff}(M)$. The main results regarding geodesic distance are contained in Section 3 and 4. In the final section 5 we derive the geodesic equations for different versions of the Sobolev metric of order s on $\operatorname{Diff}_c(\mathbb{R}^n)$ and discuss their relation to various well-known PDE's.

2. Sobolev metrics H^s with $s \in \mathbb{R}$

In this section we give definitions for Sobolev norms of fractional orders and state the properties, which we will need later to prove the vanishing geodesic distance results.

2.1. **Sobolev metrics H^s on $\mathbb{R}^n$.** For $s>0$ the Sobolev H^s-norm of an $\mathbb{R}^n$-valued function f on $\mathbb{R}^n$ is defined as

$$\|f\|^2_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}f\|^2_{L^2(\mathbb{R}^n)}\ , \tag{1}$$

where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)dx$$

and ξ is the independent variable in the frequency domain. An equivalent norm is given by

$$\|f\|^2_{\overline{H}^s(\mathbb{R}^n)} = \|f\|^2_{L^2(\mathbb{R}^n)} + \||\xi|^s\mathcal{F}f\|^2_{L^2(\mathbb{R}^n)}\ . \tag{2}$$

The fact that both norms are equivalent is based on the inequality

$$\frac{1}{C}\big(1+\sum_j|\xi_j|^s\big) \le \big(1+\sum_j|\xi_j|^2\big)^{\frac{s}{2}} \le C\big(1+\sum_j|\xi_j|^s\big)$$

holding for some constant C. For $s>1$ this says that all ℓ^s-norms on $\mathbb{R}^{n+1}$ are equivalent. But the inequality is true also for $0<s<1$, even though the expression does not define a norm on $\mathbb{R}^{n+1}$. Using any of these norms we obtain the Sobolev spaces with non-integral s

$$H^s(\mathbb{R}^n) = \{f\in L^2(\mathbb{R}^n) : \|f\|_{H^s(\mathbb{R}^n)} < \infty\}\ .$$

These spaces are also known under the name Liouville spaces or Bessel potential spaces. To make a connection with other families of function spaces, we note that the spaces $H^s(\mathbb{R}^n)$ coincide with

$$H^s(\mathbb{R}^n) = B^s_{22}(\mathbb{R}^n) = F^s_{22}(\mathbb{R}^n)$$

the Besov spaces $B^s_{22}(\mathbb{R}^n)$ and spaces of Triebel-Lizorkin type $F^s_{22}(\mathbb{R}^n)$. Definitions of these spaces and a nice introduction to the general theory of function spaces can be found in [26, section 1].

We will now collect some results about this family of spaces, which we will need in the coming sections. First a result, which states that point wise multiplication with a sufficiently smooth function is well defined.

2.2. **Theorem** (See theorem 4.2.2 in [26])**.** *Let $s>0$ and $g\in C^\infty_c(\mathbb{R}^n)$, a smooth function with compact support. Then multiplication $f\mapsto gf$ is a bounded map of $H^s(\mathbb{R}^n)$ into itself.*

We are also allowed to compose with diffeomorphisms.

2.3. **Theorem** (See theorem 4.3.2 in [26])**.** *Let $s>0$ and $\varphi\in\mathrm{Diff}_c(\mathbb{R}^n)$ be a diffeomorphism which equals the identity off some compact set. Then composition $f\mapsto f\circ\varphi$ is an isomorphic mapping of $H^s(\mathbb{R}^n)$ onto itself.*

2.4. **Sobolev metrics on Riemannian manifolds.** Following [26, section 7.2.1] we will now introduce the spaces $H^s(M)$ on a compact manifold M. Denote by $B_\varepsilon(x)$ the ball of radius ε with center x. We can choose a finite cover of M by balls $B_\varepsilon(x_\alpha)$ with ε sufficiently small, such that normal coordinates are defined in the ball $B_\varepsilon(x)$, and a partition of unity ρ_α, subordinated to this cover. Using this data we define the H^s-norm of a function f on M via

$$\|f\|^2_{H^s(M,g)} = \sum_\alpha \|(\rho_\alpha f)\circ\exp_{x_\alpha}\|^2_{H^s(\mathbb{R}^n)}$$

$$= \sum_\alpha \|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}((\rho_\alpha f)\circ \exp_{x_\alpha})\|^2_{L^2(\mathbb{R}^n)} .$$

Changing the cover or the partition of unity leads to equivalent norms, see [26, theorem 7.2.3]. For integer s we get norms which are equivalent to the Sobolev norms treated in [8, chapter 2]. The norms depend on the choice of the Riemann metric g. This dependence is worked out in detail in [8].

For vector fields we use the trivialization of the tangent bundle that is induced by the coordinate charts and define the norm in each coordinate as above. This leads to a (up to equivalence) well-defined H^s-norm on the Lie algebra $\mathfrak{X}_c(M)$ of compactly supported vector fields on M.

These definitions can be extended to manifolds of the form $M = N \times \mathbb{R}^n, n \geq 0$ in the obvious way and to manifolds of bounded geometry [26] using the results of [12, 13, 22, 7].

2.5. **Sobolev metrics on $\operatorname{Diff}_c(M)$.** Given a norm on $\mathfrak{X}_c(M)$ we can use the right-multiplication in the diffeomorphism group $\operatorname{Diff}_c(M)$ to extend this norm to a right-invariant Riemannian metric on $\operatorname{Diff}_c(M)$. In detail, given $\varphi \in \operatorname{Diff}_c(M)$ and $X, Y \in T_\varphi \operatorname{Diff}_c(M)$ we define

$$G^s_\varphi(X,Y) = \langle X\circ\varphi^{-1}, Y\circ\varphi^{-1}\rangle_{H^s(M)} .$$

In chapter 3 and in chapter 4 we are interested in questions of vanishing and non-vanishing of geodesic distance. These properties are invariant under changes to equivalent inner products, since equivalent inner products on the Lie algebra

$$\frac{1}{C}\langle X,Y\rangle_1 \leq \langle X,Y\rangle_2 \leq C\langle X,Y\rangle_1$$

imply that the geodesic distances will be equivalent metrics

$$\frac{1}{C}\operatorname{dist}_1(\varphi,\psi) \leq \operatorname{dist}_2(\varphi,\psi) \leq C\operatorname{dist}_1(\varphi,\psi) .$$

Therefore the ambiguity in the definition of the H^s-norm is of no concern to us.

In chapter 5 we will study the geodesic equations of the Sobolev metrics on $\operatorname{Diff}_c(\mathbb{R}^n)$. Equivalent norms may induce different geodesic equations. Therefore we will denote the metric that is induced by the $H^s(\mathbb{R}^n)$-norm (1) as G^s and the metric that is induced by the $\overline{H}^s(\mathbb{R}^n)$-norm (2) as $\overline{G}^s$.

3. Vanishing geodesic distance

3.1. **Theorem** (Vanishing geodesic distance). *The Sobolev metric of order s induces vanishing geodesic distance on $\operatorname{Diff}_c(M)$ if:*

- $0 \leq s < \frac{1}{2}$ *and M a Riemannian manifold that is the product of $\mathbb{R}$ with a compact manifold N, i.e., $M = \mathbb{R}\times N$.*
- $s \leq \frac{1}{2}$ *and M a Riemannian manifold that is the product of S^1 with a compact manifold N, i.e., $M = S^1 \times N$.*

This means that any two diffeomorphisms in the same connected component of $\operatorname{Diff}_c(M)$ can be connected by a path of arbitrarily short G^s-length.

Remark. *Note that this result also implies that the geodesic distance on to the homogenous space $\operatorname{Diff}(S^1)/S^1$ equipped with a homogenous metric of order $s \leq \frac{1}{2}$ vanishes. In particular the geodesic equation for the metric of order $s = \frac{1}{2}$ on this space is the modified Constantin-Lax-Majda equation, see section 5.5.*

We will prove this theorem by first constructing paths from the identity to some diffeomorphisms with arbitrary short length and then using the simplicity of the diffeomorphism group to show that any diffeomorphism can be connected to the identity with paths of arbitrary short length.

The restriction to $M \neq \mathbb{R}$ in the case $s = \frac{1}{2}$ is due to technical reasons. We believe that the result holds also in this case, however it is more difficult to construct the required vector fields in the non-compact case.

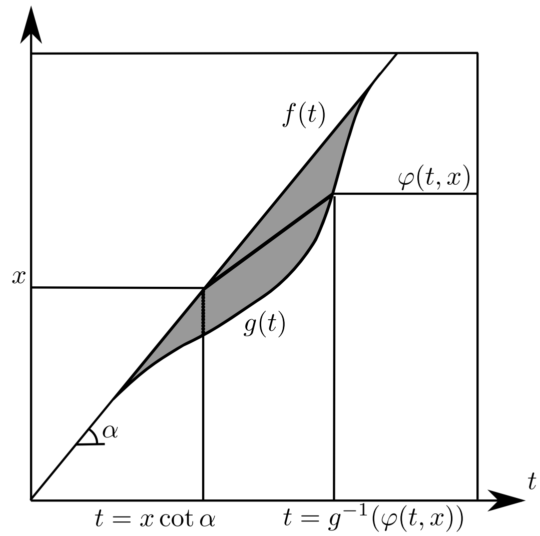

FIGURE 1. Sketch of the vector field $u(t,x)$. The gray area represents the support of u and one integral curve of $u(\cdot, x)$ is shown.

3.2. **Lemma.** *Let $\varphi \in \operatorname{Diff}_c(\mathbb{R})$ be a diffeomorphism satisfying $\varphi(x) \geq x$. Then for $0 \leq s < \frac{1}{2}$ the geodesic distance between φ and* id *with respect to the H^s-metric in $\operatorname{Diff}_c(\mathbb{R})$ is zero, i.e., φ can be connected to the identity by a path of arbitrarily short G^s-length.*

Proof. The idea of the proof is as follows. Given the diffeomorphism φ with $\varphi(x) \geq x$ we will construct a family of paths of the form

$$u(t,x) = \mathbb{1}_{[g(t),f(t)]} \star G_\varepsilon(x)$$

such that its flow $\varphi(t,x)$ satisfies $\varphi(0,x) = x$ and $\varphi(T,x) = \varphi(x)$. In a second step we will show, that when $\|f - g\|_\infty$ is sufficiently small, so is also the H^s-length of the path $\varphi(t,x)$. Here $G_\varepsilon(x) = \frac{1}{\varepsilon} G_1(\frac{x}{\varepsilon})$ is a smoothing kernel, where G_1 is a smooth bump function.

So let us construct the vector field $u(t,x)$. If we could disregard continuity, we could choose an angle $\alpha > \frac{\pi}{4}$ and set

$$f(t) = t \tan \alpha$$
$$g^{-1}(x) = x - (1 - \cot \alpha)\varphi^{-1}(x) \, .$$

The flow $\varphi(t,x)$ of the unsmoothed vector field $u(t,x) = \mathbb{1}_{[g(t),f(t)]}(x)$ satisfies

$$\varphi(t,x) = x + \int_0^t u(s, \varphi(s,x)) ds \, .$$

In this case we can write down the explicit solution, which is given by

$$\varphi(t,x) = \begin{cases} x, & t < \cot\alpha \\ t + (1-\cot\alpha)x, & x\cot\alpha \le t \le \varphi(x) - (1-\cot\alpha)x \\ \varphi(x), & t > \varphi(x) - (1-\cot\alpha)x \end{cases}$$

and we see that it satisfies the boundary conditions. We also have the relation

$$f^{-1}(x) - g^{-1}(x) = -(1-\cot\alpha)(x - \varphi^{-1}(x))$$

which implies that by choosing α sufficiently close to $\frac{\pi}{4}$ we can make $\|f-g\|_\infty$ as small as necessary. By replacing u with the smoothed vector field $\mathbb{1}_{[g(t),f(t)]} \star G_\varepsilon(x)$ we change the endpoint of the flow. However, by changing g suitably we can regain control of the endpoint. The necessary changes will be of order ε and hence we don't loose control over the difference $\|f-g\|_\infty$, which will be necessary later on.

Now we compute the norm of this vector field. Let $u(t,x)$ have the form $u(t,x) = \mathbb{1}_{[g(t),f(t)]} \star G_\varepsilon(x)$, where $f(t)$ and $g(t)$ are smooth functions which coincide off a bounded interval. To compute the H^s-norm of u, we first need to compute its Fourier transform

$$\begin{aligned}
\mathcal{F}\mathbb{1}_{[g(t),f(t)]}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{g(t)}^{f(t)} e^{i\xi x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left.\frac{e^{i\xi x}}{i\xi}\right|_{x=g(t)}^{x=f(t)} = \frac{1}{\sqrt{2\pi}} \frac{e^{i\xi f(t)} - e^{i\xi g(t)}}{i\xi} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\xi} e^{i\xi\frac{f(t)+g(t)}{2}} \frac{e^{i\xi\frac{f(t)-g(t)}{2}} - e^{-i\xi\frac{f(t)-g(t)}{2}}}{i\xi} \\
&= \frac{2}{\sqrt{2\pi}} \frac{1}{\xi} e^{i\xi\frac{f(t)+g(t)}{2}} \sin\!\left(\xi\frac{f(t)-g(t)}{2}\right) .
\end{aligned}$$

Setting $a = \frac{f(t)-g(t)}{2}$ we can now compute the norm

$$\begin{aligned}
\|\xi^s \mathcal{F}u\|_{L^2}^2 &= \|\xi^s \mathcal{F}\mathbb{1}_{[g(t),f(t)]} \mathcal{F}G_\varepsilon\|_{L^2}^2 \\
&= \int_{\mathbb{R}} \frac{2}{\pi} \frac{1}{|\xi|^{2-2s}} \sin^2(a\xi)(\mathcal{F}G_1(\varepsilon\xi))^2 d\xi \\
&\le \frac{2}{\pi} \|\mathcal{F}G_1\|_\infty^2 \int_{\mathbb{R}} \frac{\sin^2(a\xi)}{|\xi|^{2-2s}} d\xi \\
&\le \frac{2}{\pi} \|\mathcal{F}G_1\|_\infty^2 \int_{\mathbb{R}} a^{1-2s} \frac{\sin^2(\xi)}{|\xi|^{2-2s}} d\xi .
\end{aligned}$$

We get $\|u\|_{L^2}^2$ by setting $s=0$ in the above calculation. We see that for $s < \frac{1}{2}$ the H^s-norm of $u(t,.)$ is bounded by

$$\|u(t,.)\|_{H^s}^2 \le C_1|f(t)-g(t)| + C_2|f(t)-g(t)|^{1-2s} .$$

Now, putting everything together

$$\begin{aligned}
\mathrm{Len}(\varphi)^2 &= \left(\int_0^T \|u(t,.)\|_{H^s} dt\right)^2 \le T \int_0^T \|u(t,.)\|_{H^s}^2 dt \\
&\le T^2 \left(C_1\|f-g\|_\infty + C_2\|f-g\|_\infty^{1-2s}\right) .
\end{aligned}$$

Since the geodesic length is defined as the infimum over all paths and since we have shown in the first part of the proof that by choosing the angle α and the smoothing factor ε, we can control the norm $\|f-g\|_\infty$, the proof is complete. $\square$

In the case $s = \frac{1}{2}$ we will work on the circle S^1. Using a construction similar to that in Lemma 3.2 we will construct arbitrary short paths from the identity to the shift $\varphi(x) = x+1$. The following lemma supplies us with functions that have small $H^{1/2}$-norm and large L^∞-norm at the same time. We cannot use step functions as in the above proof, because they are not in $H^{1/2}(S^1)$.

3.3. **Lemma.** *Let $\psi(x)$ be a non-negative, compactly supported C^∞-function on $\mathbb{R}$ and $(b_j)_{j=0}^\infty$ a non-increasing sequence of non-negative numbers, with $\sum b_j^2 < \infty$. Then the $H^{1/2}$-norm of the function $f(x) := \sum b_j \psi(2^j x)$ is bounded by*

$$\|f\|^2_{H^{1/2}} \leq C \sum_{j=0}^{\infty} b_j^2 \,.$$

Proof. This result is shown in step 4 of the proof of theorem 13.2 in [27]. □

The main difference between Lemma 3.2 and the following Lemma is that on S^1 we don't have to worry about the diffeomorphisms having compact support. On $\mathbb{R}$ the diffeomorphism $\varphi(x) = x+1$ is not element of $\mathrm{Diff}_c(\mathbb{R})$ and we would have to replace it by $\varphi(x) = x + c(x)$, where $c(x)$ is some function with compact support. This makes working on the circle much easier.

3.4. **Lemma.** *Let $\varphi \in \mathrm{Diff}(S^1)$ be the shift by 1, i.e. $\varphi(x) = x+1$. Then the geodesic distance between φ and* id *with respect to the $H^{1/2}$-metric in $\mathrm{Diff}(S^1)$ is zero.*

Proof. We will prove the lemma by constructing a sequence of vector fields with arbitrary small $H^{1/2}$-norms, whose flows at time $t = T_{\text{end}}$ will be $\varphi(T_{\text{end}}, x) = x+1$. First we apply Lemma 3.3 with $b_j = \frac{1}{N}$ for $j = 0, \ldots, N-1$ and 0 otherwise. By doing so we obtain $\|f\|^2_{H^{1/2}} \leq C\frac{1}{N}$, while $\|f\|_\infty = 1$. For the basic function $\psi(x)$ we choose $\psi(x) = e^{\frac{1}{1-|x|^2}}$. Note that $\mathrm{supp}(\psi) \subseteq [-1,1]$ and that ψ is concave in a neighborhood around 0. Since each f is a finite sum of $\psi(2^j x)$, these properties hold also for f.

We define the vector field

$$u(t,x) = \lambda f(t-x) \quad \text{with} \quad 0 \leq \lambda < 1$$

for $t \in [0, T_{\text{end}}]$, where T_{end} will be specified later. The energy of this path is bounded by

$$E(u) = \int_0^{T_{\text{end}}} \|u(t,.)\|^2_{H^{1/2}} \,\mathrm{d}t \leq C T_{\text{end}} \frac{1}{N}$$

and hence can be made as small as necessary. It remains to show that the flow of this vector field at time $t = T_{\text{end}}$ is indeed $\varphi(T_{\text{end}}, x) = x+1$.

We do this in several steps. First we consider this vector field defined on all of $\mathbb{R}$ with time going from $-\infty$ to ∞. The initial condition for the flow is $\varphi(-\infty, x) = x$. Since $u(t,x)$ has compact support in x, this doesn't cause any analytical problems. As long as $\lambda < 1$ each integral curve of u will leave the support of u after finite time. Therefore we can consider $\varphi(\infty, x)$ to be the endpoint of the flow. Next we will establish that $\varphi(\infty, x) = x + S$ is a uniform shift, that is independent of x. Then we show that by appropriately choosing λ we can control the amount of shifting, in particular we can always obtain $S = 1$. Then we find bounds for the time each integral curve spends in the support of u. By showing that this time is only dependent on S, but not on the specific form of f or ψ, we will know that T_{end}

doesn't grow larger as we let $N \to \infty$. In the last step we go back to the circle, define T_{end}, start the flow at time $t = 0$ and show that the resulting flow is a shift by 1 at time T_{end}. This will conclude the proof.

The flow $\varphi(t, x)$ of $u(t, x)$ is given by the equation

$$\partial_t \varphi(t, x) = u(t, \varphi(t, x))$$

with the initial condition $\varphi(-\infty, x) = x$. Define the function $a_x(t) = t - \varphi(t, x)$. Because $\partial_t a_x(t) = 1 - \lambda f(a_x(t)) > 0$ the function $a_x(.)$ is a diffeomorphism in t for each fixed x. Since $\operatorname{supp}(f) \subseteq [-1, 1]$, we have $\partial_t \varphi(t, x) \neq 0$ only for $t \in [a_x^{-1}(-1), a_x^{-1}(1)]$. Let us define $T_{\text{shift}} = a_x^{-1}(1) - a_x^{-1}(-1)$ to be the time necessary for the flow to pass through the vector field u.

Claim A. *T_{shift} is independent of x.*

This follows from the following symmetries of the flow $\varphi(t, x)$ and the map $a_x(t)$. We have

$$\varphi(t, x) = \varphi(t - (x - y), y) + x - y$$

and

$$a_x(t) = a_y(t - (x - y)) \; .$$

To prove the first identity assume $x > y$ and note that at time $t_0 = y - 1$ we have

$$\varphi(y - 1, x) = x = x + \varphi(y - 1 - (x - y), y) - y$$

since at time $y - 1 - (x - y)$ the flow $\varphi(t, y)$ still equals y. Now differentiate to see that both functions satisfy the same ODE. The second identity is an immediate consequence of the first one. To prove the claim that T_{shift} is independent of x, we will show that

$$\partial_x a_x^{-1}(t) = 1 \; .$$

Start with $a_x(a_x^{-1}(t)) = t$, use the symmetry relation $a_0(a_x^{-1}(t) - x) = t$ and differentiate with respect to x to obtain

$$\partial_t a_0(a_x^{-1}(t) - x)(\partial_x a_x^{-1}(t) - 1) = 0 \; ,$$

which concludes the proof of the claim.

For each x, the flow of the vector field performs a shift $[a_x^{-1}(-1), a_x^{-1}(1)]$ given by

$$\varphi(\infty, x) = x + \int_{a_x^{-1}(-1)}^{a_x^{-1}(1)} \lambda f(t - \varphi(t, x)) \, \mathrm{d}t = x + \int_{-1}^{1} \frac{\lambda f(t)}{1 - \lambda f(t)} \, \mathrm{d}t \; .$$

Define

$$I(\lambda) = \int_{-1}^{1} \frac{\lambda f(t)}{1 - \lambda f(t)} \, \mathrm{d}t$$

to be the amount of shifting that is taking place as a function of λ. Note that $I(\lambda)$ is smooth in $\lambda \in [0, 1)$ with $I(0) = 0$ and $I'(0) > 0$. We claim that we can always choose λ close enough to 1, to obtain any shift necessary.

Claim B. *$I : [0, 1) \to \mathbb{R}_{\geq 0}$ is a diffeomorphism*

Obviously, $\partial_\lambda I(\lambda) > 0$ and $I(0) = 0$. It remains to show that $\lim_{\lambda \to 1} I(\lambda) = \infty$. Each f that we choose in our construction is concave in some small neighborhood around 0. So choose $a > 0$, such that for $t \in [0, a]$ we have $f(t) \geq \frac{1}{2}$ and $f(t) \geq 1 - ct$ for some constant c. Then we can estimate the integral by

$$I(\lambda) \geq \int_0^a \frac{\lambda f(t)}{1 - \lambda f(t)} \, \mathrm{d}t \geq \frac{\lambda}{2} \int_0^a \frac{1}{1 - \lambda + \lambda c t} \, \mathrm{d}t$$

$$\geq \frac{\lambda}{2} \log(1 - \lambda + \lambda ct)\Big|_{t=0}^{t=a}$$
$$\geq \frac{\lambda}{2} \log\left(1 + \frac{\lambda ca}{1-\lambda}\right) .$$

From this we can see that $I(\lambda)$ grows towards infinity.

Note that a similar calculation shows that $\partial_\lambda I(\lambda) > a$ for some $a \geq 0$.

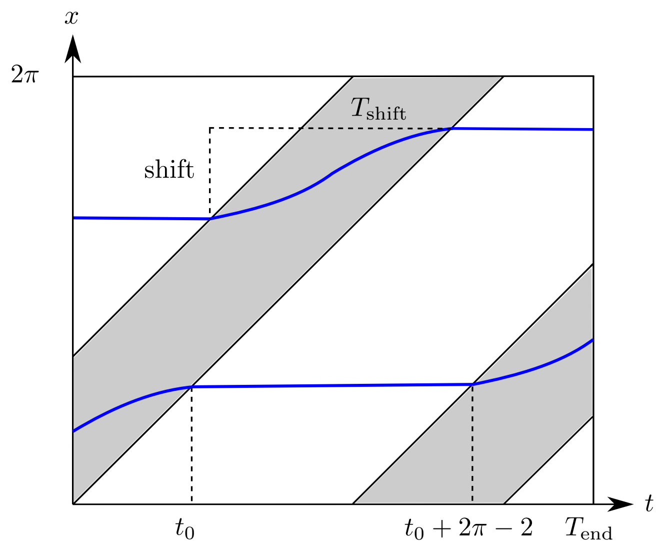

FIGURE 2. Sketch of the vector field $u(t, x)$. The gray area represents the support of u and the blue curves are integral curves of $u(\cdot, x)$.

Claim C. *We have control over the time necessary to induce a shift $I(\lambda)$, via $T_{\text{shift}} = 2 + I(\lambda)$.*

We had above $T_{\text{shift}} = a_x^{-1}(1) - a_x^{-1}(-1)$ and also $\partial_t a_x^{-1}(t) = \frac{1}{1-\lambda f(t)}$. Hence

$$T_{\text{shift}} = \int_{-1}^{1} \frac{1}{1-\lambda f(t)}\, \mathrm{d}t = \int_{-1}^{1} 1 + \frac{\lambda f(t)}{1-\lambda f(t)}\, \mathrm{d}t = 2 + I(\lambda) .$$

This concludes the proof of the claim.

Now we choose λ such that $I(\lambda) = 1$ and we define the flow $\varphi(t, x)$ on the circle with period 2π for time from 0 to $T_{\text{end}} = T_{\text{shift}} + 2\pi - 2$.

Claim D. *The endpoint of the resulting flow is a constant shift*

$$\varphi(T_{\text{end}}, x) = x + 1 .$$

We have to consider two cases. First take a point x, such that $x \notin \operatorname{supp}(f)$. W.l.o.g. we assume that $\operatorname{supp}(f)$ is the interval $[0, 2]$ starting at 0, which implies that in this case $x > 2$. Then x meets the vector field at time $t_0 = x - 2$ and leaves it again at time $t_1 = x - 2 + T_{\text{shift}}$ after being shifted by one. It would meet the vector field again at time $t_2 = x - 2 + T_{\text{shift}} + 2\pi - 2$, but because $x - 2 > 0$ we have $t_2 > T_{\text{end}}$ and hence the point doesn't meet the vector field again.

Now take a point $0 \leq x \leq 2$. This point starts in the vector field, leaves it at some time $t_0 < T_{\text{shift}}$ and meets the vector field again at time $t_0 + 2\pi - 2$. It then

stays in the vector field until the T_{end} since $T_{\text{end}} < t_0 + 2\pi - 2 + T_{\text{shift}}$. Altogether the point has spent time $t_0 + T_{\text{end}} - (t_0 + 2\pi - 2) = T_{\text{shift}}$ in the vector field, so it has also been shifted by 1. This concludes the proof. $\square$

Proof of theorem 3.1. To prove vanishing of the geodesic distance we will follow the idea of [18], where it was proven that the geodesic distance vanishes on $\operatorname{Diff}_c(M)$ for the right-invariant L^2-metric. To be precise, we only consider the connected component $\operatorname{Diff}_0(M)$ of Id, i.e. those diffeomorphisms of $\operatorname{Diff}_c(M)$, for which there exist at least one path, joining them to the identity. Let us denote by $\operatorname{Diff}_c(M)^{L=0}$ the set of all diffeomorphisms φ that can be reached from the identity by curves of arbitrarily short length, i.e., for each $\varepsilon > 0$ there exists a curve from the identity to φ with length smaller than ε.

In the following we will show that $\operatorname{Diff}_c(M)^{L=0}$ is a non-trivial normal subgroup of $\operatorname{Diff}_c(M)$ (and $\operatorname{Diff}_0(M)$). It was shown in [9, 25, 14, 15] that $\operatorname{Diff}_0(M)$ is a simple group, i.e. only the identity and the whole group are normal subgroups of $\operatorname{Diff}_0(M)$. From this it follows that $\operatorname{Diff}_c(M)^{L=0}$ is the whole connected component $\operatorname{Diff}_0(M)$. In other words, every diffeomorphism that can be connected to the identity, can be connected via a path of arbitrary short length.

Claim A. $\operatorname{Diff}_0(M)^{L=0}$ *is a normal subgroup of* $\operatorname{Diff}_0(M)$.
Given a diffeomorphism $\psi \in \operatorname{Diff}_0(M)$, we can choose a partition of unity τ_j such that normal coordinates centered at $x_j \in M$ are defined on $\operatorname{supp}(\tau_j)$ and such that normal coordinates centered at $\psi(x_j)$ are defined on $\psi(\operatorname{supp}(\tau_j))$. Then we can define $\psi_j = \exp^{-1}_{\psi(x_j)} \circ \psi \circ \exp_{x_j}$. For $\varphi_1 \in \operatorname{Diff}_0(M)^{L=0}$ we choose a curve $t \mapsto \varphi(t, \cdot)$ from the identity to φ_1 with length less than ε. Let $u = \varphi_t \circ \varphi^{-1}$. Then

$$\begin{aligned}
\operatorname{Len}(\psi^{-1} \circ \varphi \circ \psi) &\leq C_1(\tau) \int_0^1 \|(T\psi^{-1} \circ \varphi_t \circ \psi) \circ (\psi^{-1} \circ \varphi \circ \psi)^{-1}\|_{H^s(M,\tau)} dt \\
&= C_1(\tau) \int_0^1 \|T\psi^{-1} \circ u \circ \psi\|_{H^s(M,\tau)} dt \\
&= C_1(\tau) \int_0^1 \sqrt{\sum_j \|\exp^*_{x_j}(\tau_j . T\psi^{-1} \circ u \circ \psi)\|^2_{H^s(\mathbb{R}^n)}} dt \\
&= C_1(\tau) \int_0^1 \sqrt{\sum_j \|T\psi_j^{-1} . (\exp^*_{\psi(x_j)}(\tau_j \circ \psi^{-1} . u)) \circ \psi_j\|^2_{H^s(\mathbb{R}^n)}} dt \\
&\leq C_2(\psi, \tau) \int_0^1 \sqrt{\sum_j \|(\exp^*_{\psi(x_j)}(\tau_j \circ \psi^{-1} . u))\|^2_{H^s(\mathbb{R}^n)}} dt \\
&= C_2(\psi, \tau) \int_0^1 \|u\|_{H^s(M, \tau \circ \psi^{-1})} dt \leq C_3(\psi, \tau) \operatorname{Len}(\varphi) .
\end{aligned}$$

Here we used that all partitions of unity τ induce equivalent norms $H^s(M, \tau)$ and that for $h \in C^\infty(M)$ and $\psi \in \operatorname{Diff}_c(M)$ point wise multiplication $f \mapsto h.f$ and composition $f \mapsto f \circ \psi$ are bounded linear operators on $H^s(M)$, as noted in theorems 2.2 and 2.3.

Claim B. $\operatorname{Diff}_0(M)^{L=0}$ *is a nontrivial subgroup of* $\operatorname{Diff}_0(M)$.
For a one-dimensional manifold M the non-triviality of $\operatorname{Diff}_0(M)^{L=0}$ under appropriate conditions on s is shown in lemmma 3.2 and lemma 3.4. The higher dimensional cases treated here are quite obvious, since we can endow $M = S^1 \times N$

or $M = \mathbb{R} \times N$ with a product metric and use the paths obtained in the corresponding one-dimensional case. To spell this out let $\varphi_\varepsilon(t,x)$ be a family of paths on $\operatorname{Diff}_c(\mathbb{R})$ or $\operatorname{Diff}(S^1)$ indexed by ε such that each path φ_ε connects the identity to some diffeomorphism $\varphi_\varepsilon(T,x) = \varphi(x)$ and has H^s-length smaller than ε. Then $\psi_\varepsilon(t,(x,y)) = (\varphi_\varepsilon(t,x), y)$ defines a family of paths on $\operatorname{Diff}_c(M)$ connecting Id_M to $(\varphi, \operatorname{Id}_N)$ with arbitrarily short H^s-length.

□

4. Positive Geodesic Distance

4.1. **Theorem** (Positive geodesic distance). *For* $\dim(M) = 1$ *the Sobolev-norm of order s induces positive geodesic distance on* $\operatorname{Diff}_c(M)$ *if* $s > \frac{1}{2}$. *For* $\dim(M) \geq 2$ *it induces positive geodesic distance if* $s \geq 1$.

Proof. By the definition of the Sobolev metric it suffices to show the result for $\operatorname{Diff}_c(\mathbb{R}^n)$.

For the case $n = 1$ let $\varphi_0, \varphi_1 \in \operatorname{Diff}_c(\mathbb{R})$ with $\varphi_0(x) \neq \varphi_1(x)$ for some $x \in \mathbb{R}$. For any path $\varphi(t,\cdot)$, with $\varphi(0,\cdot) = \varphi_0$ and $\varphi(1,\cdot) = \varphi_1$ we have

$$\begin{aligned}
0 \neq |\varphi_1(x) - \varphi_0(x)| &= \left| \int_0^1 \varphi_t(t,x) dt \right| = \left| \int_0^1 u(t, \varphi(t,x)) dt \right| \\
&\leq \int_0^1 |u(t,\varphi(t,x))| \, dt \leq \int_0^1 \|u(t,\cdot)\|_\infty dt \leq \int_0^1 \|u(t,\cdot)\|_{C^{0,s-1/2}} dt \\
&\leq \int_0^1 \|u(t,\cdot)\|_{H^s} dt \; .
\end{aligned}$$

In the last step, we used the Sobolev embedding theorem, see [26] for example. The case $\dim(M) \geq 2$ follows from [18, Theorem 5.7]. □

5. The Geodesic Equation on $\operatorname{Diff}_c(\mathbb{R}^n)$

In the upcoming parts we want to calculate the geodesic equation for the two equivalent Sobolev norms on $\operatorname{Diff}_c(\mathbb{R}^n)$.

5.1. **The general setting.** According to [1], see [17, section 3] for a presentation directly applicable here, we have: For any right invariant metric G on a regular infinite dimensional Lie group, the geodesic equation reads as

$$u_t = -\operatorname{ad}(u)^\top u \; .$$

Here $\operatorname{ad}(u)^\top$ denotes the adjoint of the adjoint representation ad, which is given by $\langle \operatorname{ad}(v)^\top u, w \rangle_G := \langle u, \operatorname{ad}_v w \rangle_G$. Note that for $\operatorname{Diff}_c(\mathbb{R}^n)$ we have $\operatorname{ad}_v w = -[v,w]$ for $v, w \in \mathfrak{X}_c(\mathbb{R}^n)$. The sectional curvature (for orthonormal u, v) at the identity is then given by the formula

$$\langle R(u,v)v, u \rangle_G = \frac{1}{4} \|\beta(u)v - \beta(v)u - \operatorname{ad}(u)v\|_G^2 + \langle [\beta(u), \beta(v)]u, v \rangle_G$$

where $\beta(u)v := \operatorname{ad}(u)^\top v + \operatorname{ad}(u)v$. This last expression is from [16, section 2.6]. The Jacobi equation for a right trivialized Jacobi field y along a geodesic with right trivialized velocity field u (which satisfies the geodesic equation) is derived in [20, 3.4 and 3.5] as:

$$y_{tt} = [\operatorname{ad}(y)^\top + \operatorname{ad}(y), \operatorname{ad}(u)^\top] u - \operatorname{ad}(u)^\top y_t - \operatorname{ad}(y_t)^\top u + \operatorname{ad}(u) y_t \, .$$

This will allow us to write down the curvature and the Jacobi equations for all metrics that we will treat below. Since this leads to complicated formulas we will not spell this out.

5.2. **Theorem.** *Let $A : \mathfrak{X}_c(\mathbb{R}^n) \to \mathfrak{X}_c(\mathbb{R}^n)$ be an elliptic, scalar (pseudo)-differential operator that is positive and self-adjoint with respect to the L^2-metric. Then A induces a metric on $\operatorname{Diff}_c(\mathbb{R}^n)$ in the following way:*

$$G^A_\varphi(X,Y) := \langle A(X\circ\varphi^{-1}), Y\circ\varphi^{-1}\rangle_{L^2(\mathbb{R}^n)} .$$

The geodesic equation with respect to the G^A-metric is then given by

$$Au^k_t = -\sum_{i=1}^n \left(Au^i(\partial_k u^i) + (A(\partial_i u^k).u^i + Au^k.(\partial_i u^i))\right).$$

Equivalently it can be written in terms of the momentum $m = Au$:

$$m^k_t = -\sum_{i=1}^n \left(m^i(\partial_k u^i) + ((\partial_i m^k).u^i + m^k.(\partial_i u^i))\right), \quad u^k = A^{-1}m^k.$$

Proof. For $u, v, w \in \mathfrak{X}_c(\mathbb{R}^n)$ we calculate

$$\begin{aligned}
\langle u, -[v,w]\rangle_{G^A} &= \int_{\mathbb{R}^n} \langle Au, -[v,w]\rangle_{\mathbb{R}^n}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{k=1}^n Au^k \sum_{i=1}^n \left((\partial_i v^k)w^i - v^i(\partial_i w^k)\right)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{k=1}^n \sum_{i=1}^n Au^k(\partial_i v^k)w^i + \partial_i(Au^k.v^i)\partial_i w^k\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{i=1}^n \sum_{k=1}^n Au^i(\partial_k v^i)w^k + \sum_{k=1}^n \sum_{i=1}^n (A(\partial_i u^k).v^i + Au^k.(\partial_i v^i))w^k\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{k=1}^n w^k A\left(\operatorname{ad}(v)^\top u\right)^k\,\mathrm{d}x = \langle \operatorname{ad}(v)^\top u, w\rangle_{G^A} .
\end{aligned}$$

$$\operatorname{ad}(v)^\top u = \sum_{i=1}^n A^{-1}\left(Au^i(\partial_k u^i) + (A(\partial_i u^k).u^i + Au^k.(\partial_i u^i))\right) . \qquad \square$$

5.3. **Theorem** (Geodesic equation for the Sobolev metric G^s)**.** *The operator*

$$A_s : H^{k+2s}(\mathbb{R}^n) \to H^k(\mathbb{R}^n), \quad u(x) \mapsto \left(\mathcal{F}^{-1}(1+|\xi|^2)^s\mathcal{F}u\right)(x)$$

induces the Sobolev metric G^s of order s on $\operatorname{Diff}_c(\mathbb{R}^n)$. The geodesic equation for this metric reads as:

$$\begin{aligned}
m^k_t &= -\sum_{i=1}^n \left(m^i(\partial_k u^i) + ((\partial_i m^k).u^i + m^k.(\partial_i u^i))\right), \\
u^k &= \begin{cases} (2\pi)^{\frac{n}{2}}\,\frac{2^{1-s}|.|^{s-\frac{n}{2}}}{\Gamma(s)} K_{s-\frac{n}{2}}(|\cdot|)\star m^k, & s > \frac{n-1}{4} \\ (2\pi)^{\frac{n}{2}}|\cdot|^{1-\frac{n}{2}}\int_0^\infty J_{\frac{n}{2}-1}(r.|\cdot|)\frac{r^{\frac{n}{2}}}{(1+r^2)^s}\,\mathrm{d}r\star m^k, & s \le \frac{n-1}{4} . \end{cases}
\end{aligned}$$

Here $J_{n/2-1}$ denotes the Bessel function of the first kind, which is given by

$$J_\alpha(r) = \frac{1}{\pi}\int_0^\pi \cos(\alpha t - r\sin t)\,dt - \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-r\sinh(t)-\alpha t}\,dt ,$$

and $K_{s-\frac{n}{2}}$ denotes the modified Bessel function of second kind, which is given by

$$K_\nu(r) = \frac{\Gamma(\nu+\frac{1}{2})(2r)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\cos t}{(t^2+r^2)^{\nu+\frac{1}{2}}}\,\mathrm{d}t\ .$$

Proof. The operator $A_s u = \mathcal{F}^{-1}(1+|\xi|^2)^s \mathcal{F}u$ is an elliptic, scalar (pseudo)-differential operator that is positive and self-adjoint with respect to the L^2-metric. In particular it is a linear isomorphism from $H^{k+2s}(\mathbb{R}^n) \to H^k(\mathbb{R}^n)$. By theorem 5.2 it remains to calculate the operator A_s^{-1}. This can be done as follows:

$$A_s^{-1}m(x) = \mathcal{F}^{-1}(1+|\xi|^2)^{-s}\mathcal{F}m(x) = (2\pi)^{\frac{n}{2}}\mathcal{F}^{-1}\left((1+|\xi|^2)^{-s}\right)\star m(x)\ .$$

Let f be a radial symmetric function on $\mathbb{R}^n$, i.e., $f(\xi) = f(|\xi|)$. Then we have

$$\mathcal{F}^{-1}(f)(x) = |x|^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(r.|x|).f(r).r^{\frac{n}{2}}\,dr\ .$$

For the function $f(\xi) = (1+|\xi|^2)^{-s}$ this yields:

$$\mathcal{F}^{-1}\left((1+|\xi|^2)^{-s}\right) = |x|^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(r.|x|)\frac{r^{\frac{n}{2}}}{(1+r^2)^s}\,\mathrm{d}r\ .$$

See for example the books [23, 24] for more details about Fourier transformation of radial symmetric functions. For $s > \frac{n-1}{4}$ the last integral converges and we have

$$\mathcal{F}^{-1}\left((1+|\xi|^2)^{-s}\right) = \frac{2^{1-s}|x|^{s-\frac{n}{2}}}{\Gamma(s)} K_{s-\frac{n}{2}}(|x|)\ . \qquad \square$$

An immediate consequence of the above analysis is the geodesic equation for the equivalent Sobolev-metric $\overline{G}^s$.

5.4. **Theorem** (Geodesic equation for the Sobolev metric $\overline{G}^s$)**.** *The operator*

$$\overline{A}_s : H^{k+2s}(\mathbb{R}^n) \to H^k(\mathbb{R}^n), \quad u(x) \mapsto \left(\mathcal{F}^{-1}(1+|\xi|^{2s})\mathcal{F}u\right)(x)$$

induces the Sobolev metric $\overline{G}^s$ of order s on $\operatorname{Diff}_c(\mathbb{R}^n)$. The geodesic equation for this metric reads as:

$$m_t^k = -\sum_{i=1}^n \left(m^i(\partial_k u^i) + ((\partial_i m^k).u^i + m^k.(\partial_i u^i))\right)\ ,$$

$$u^k = (2\pi)^{\frac{n}{2}} |\cdot|^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(r.|\cdot|)\frac{r^{\frac{n}{2}}}{(1+r^{2s})}\,\mathrm{d}r \star m^k\ .$$

5.5. **The Geodesic Equation in dimension one.** For the $\overline{G}^s$-metric on $\operatorname{Diff}_c(\mathbb{R})$ or $\operatorname{Diff}(S^1)$ the above expression for the geodesic equation simplifies to

$$m_t = -2u_x m - u m_x, \quad u = (2\pi)^{\frac{1}{2}} \int_0^\infty J_{-\frac{1}{2}}(r.|\cdot|)\frac{r^{\frac{1}{2}}}{(1+r^{2s})}\,\mathrm{d}r \star m\ .$$

For $s = k \in \mathbb{N}$ we can rewrite this equation as:

$$m_t = -2u_x m - u m_x, \quad m = u + \partial_x^{2k} u\ ,$$

where $m(t,x)$ is the momentum corresponding to the velocity $u(t,x)$. In the case $s = 0$ this becomes the inviscid Burger equation

$$u_t = -3u_x u\ ,$$

for $s = 1$ it is the Camassa Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + u u_{xxx}\ ,$$

and for $s > 1$ they are related to the higher order Camassa Holm equations, see [5].

To study the Sobolev metric $\overline{G}^s$, for $s = k + \frac{1}{2}$ we introduce the Hilbert transform, which is given by

$$\mathcal{F}(\mathcal{H}f)(\xi) = -i\,\mathrm{sgn}(\xi)\mathcal{F}f(\xi)\ .$$

Using this we can write the metric $\overline{G}^{k+\frac{1}{2}}$ in the form

$$\overline{G}^{k+\frac{1}{2}}(u,v) = \int_{\mathbb{R}} (u + \mathcal{H}\partial_x^{2k+1}u)v\,\mathrm{d}x = \int_{\mathbb{R}} (u + \partial_x^{2k+1}(\mathcal{H}u))v\,\mathrm{d}x\ .$$

The geodesic equation is then given by

$$m_t = -2u_x m - u m_x, \quad m = u + \mathcal{H}\partial_x^{2k+1}u\ .$$

If we pass to the homogenous space $\mathrm{Diff}(S^1)/S^1$ and consider the homogenous Sobolev metric of order one half the geodesic equation reduces to

$$m_t = -2u_x m - u m_x, \quad m = \mathcal{H}u_x\ ,$$

which is the modified Constantin-Lax-Majda equation, see [28]. If we consider the homogenous Sobolev metric of order one the resulting geodesic equation

$$u_{xxt} = -2u_x u_{xx} - u u_{xxx}$$

is the Hunter-Saxton equation.

6. Conclusions.

In this paper we have provided a partial answer to the problem of vanishing geodesic distance for Sobolev-type metrics on diffeomorphism groups. Some cases remain open.

Conjecture. For $M = \mathbb{R}$ and $s = \frac{1}{2}$ the geodesic distance vanishes.

We believe that a similar construction as in Lemma 3.4 can be used to construct paths of arbitrary short length. However, since the diffeomorphisms need to have compact support, we need to adapt the construction to reach a diffeomorphism of the form $\varphi(x) = x + c(x)$ as in Lemma 3.2. The difficulty lies in constructing a vector field, whose flow at time $t = 1$ we can control.

The more interesting question is about the behaviour for arbitrary Riemannian manifolds of bounded geometry and higher dimension:

Conjecture. The geodesic distance vanishes on the space $\mathrm{Diff}_c(M)$ for $0 \le s \le \frac{1}{2}$, for M an arbitrary manifold of bounded geometry.

For $0 \le s \le \frac{1}{2}$ we believe that it is possible to adapt the method of [18] to show that there exists a diffeomorphism that can be connected to the identity by paths of arbitrary short length. The simplicity of the diffeomorphism group then concludes the proof.

Conjecture. The geodesic distance is positive for $s > \frac{1}{2}$.

The proof of Theorem 4.1 establishing positivity of the geodesic distance for $\dim(M) = 1$ and $s > \frac{1}{2}$ is based on the Sobolev embedding theorem, which holds for s greater than the critical index $\dim(M)/2$. The argument generalizes to higher dimensions, but in $\dim(M) \ge 2$ the result [18, Theorem 5.7] is stronger. Namely, it is shown that the geodesic distance is positive for $s \ge 1$ in all dimensions. To prove the conjecture it remains to improve the bound $s \ge 1$ to $s > \frac{1}{2}$ for $\dim(M) \ge 2$.

The proof of Theorem 3.1 provides a hint that this can be done and that the bound $s > \frac{1}{2}$ is optimal. The idea of the proof in dimension one is to compress the space to a point and to move this point around. This results in a path of

diffeomorphisms with short H^s-length because there are functions with small H^s-norm that are nevertheless large at some point. The obvious generalization to higher dimensions is to compress the space to a set of codimension one and to move this set around. Again, this should result in a path of short H^s-length when there are functions with small H^s-norm that are large at a set of codimension one. This is the case exactly for $s \leq \frac{1}{2}$.

Another interesting question is whether our results carry over to the Virasoro-Bott group. In [2] it was shown that the right invariant L^2-metric on the Virasoro-Bott group has vanishing geodesic distance. The key to the proof in [2] is to control the central cocycle along a curve of diffeomorphisms. This is done by expressing the cocycle in terms of the diffeomorphism and its derivatives. In contrast to this, all derivatives in the constructions of the present work are left-trivialized. We believe that this is not a serious obstacle and that the results of this paper can be extended to Sobolev metrics of fractional order on the Virasoro-Bott group.

References

[1] V. I. Arnold. Sur la géometrie différentielle des groupes de lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier*, 16:319–361, 1966.

[2] M. Bauer, M. Bruveris, P. Harms, and P.W. Michor. Vanishing geodesic distance for the riemannian metric with geodesic equation the KdV-equation. *Ann. Global Analysis Geom.* 41, 4 (2012) 461-472.

[3] M. Bauer, P. Harms, and P. W. Michor. Sobolev metrics on shape space of surfaces. *Journal of Geometric Mechanics* 3, 4 (2011), 389-438.

[4] A. Constantin, T. Kappeler, B. Kolev, and P. Topalov. On geodesic exponential maps of the Virasoro group. *Ann. Global Anal. Geom.*, 31(2):155–180, 2007.

[5] Adrian Constantin and Boris Kolev. Geodesic flow on the diffeomorphism group of the circle. *Comment. Math. Helv.*, 78(4):787–804, 2003.

[6] David G. Ebin and Jerrold Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. of Math. (2)*, 92:102–163, 1970.

[7] Jürgen Eichhorn. The boundedness of connection coefficients and their derivatives. *Math. Nachr.*, 152:145–158, 1991.

[8] Jürgen Eichhorn. *Global Analysis on Open Manifolds*. Nova Science Publishers Inc., New York, 2007.

[9] D. B. A. Epstein. The simplicity of certain groups of homeomorphisms. *Compositio Math.*, 22:165–173, 1970.

[10] Joachim Escher, Boris Kolev, and Marcus Wunsch. The geometry of a vorticity model equation. arXiv:1010.4844v1 [math.AP] 23 Oct 2010.

[11] François Gay-Balmaz. Well-posedness of higher dimensional Camassa-Holm equations. *Bull. Transilv. Univ. Braşov Ser. III*, 2(51):55–58, 2009.

[12] R. E. Greene. Complete metrics of bounded curvature on noncompact manifolds. *Arch. Math.*, 31(1):89–95, 1978.

[13] Yu. A. Kordyukov. L^p-theory of elliptic differential operators on manifolds of bounded geometry. *Acta Appl. Math.*, 23(3):223–260, 1991.

[14] John N. Mather. Commutators of diffeomorphisms. *Comment. Math. Helv.*, 49:512–528, 1974.

[15] John N. Mather. Commutators of diffeomorphisms. II. *Comment. Math. Helv.*, 50:33–40, 1975.

[16] Mario Micheli, Peter W. Michor, and David Mumford. Sobolev curvature of $\mathrm{Diff}(\mathbb{R}^n)$ and of some of its Chow quotients. *In preparation.*

[17] Peter W. Michor. Some geometric evolution equations arising as geodesic equations on groups of diffeomorphisms including the Hamiltonian approach. In *Phase space analysis of partial differential equations*, volume 69 of *Progr. Nonlinear Differential Equations Appl.*, pages 133–215. Birkhäuser Boston, 2006.

[18] Peter W. Michor and David Mumford. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms. *Doc. Math.*, 10:217–245 (electronic), 2005.

[19] Peter W. Michor and David Mumford. Riemannian geometries on spaces of plane curves. *J. Eur. Math. Soc. (JEMS)*, 8:1–48, 2006.
[20] P.W. Michor and T. Ratiu. Geometry of the Virasoro-Bott group. *J. Lie Theory*, 8:293–309, 1998.
[21] V. Y. Ovsienko and B. A. Khesin. Korteweg–de Vries superequations as an Euler equation. *Funct. Anal. Appl.*, 21:329–331, 1987.
[22] M. A. Shubin. Spectral theory of elliptic operators on noncompact manifolds. *Astérisque*, 207(5):35–108, 1992. Méthodes semi-classiques, Vol. 1 (Nantes, 1991).
[23] Elias M. Stein and Rami Shakarchi. *Fourier Analysis*, volume 1 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2003. An introduction.
[24] Elias M. Stein and Guido Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.
[25] William Thurston. Foliations and groups of diffeomorphisms. *Bull. Amer. Math. Soc.*, 80:304–307, 1974.
[26] Hans Triebel. *Theory of Function Spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.
[27] Hans Triebel. *The Structure of Functions*, volume 97 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2001.
[28] Marcus Wunsch. On the geodesic flow on the group of diffeomorphisms of the circle with a fractional Sobolev right-invariant metric. *J. Nonlinear Math. Phys.*, 17(1):7–11, 2010.

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